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LETTER TO THE EDITOR

**Weight conservation and quantum group construction of the braid group representation**

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**Abstract.** It is proved that the braid group representation constructed in terms of the quantum group can be covered by that obtained from the proposal of weight conservation by the extended Kauffman diagram technique. The non-standard braid group representations associated with SU(2) are obtained by adding the terms of weight conservation to the standard universal *R*-matrix.

At present the theories of braid group representation (BGR) draw much attention in mathematical physics [1] because of their key roles for the solvable models and low dimensional field theories through the Yang-Baxter equation (YBE). The standard BGRs or the *R*-matrices have been constructed in terms of the quantum groups (QGs) as *q*-analogues of the universal enveloping algebras (QUEAs) of Lie algebras in mathematics [2-4]. Recently, the Kauffman diagram technique (KDT) [5] associated with SU(2) has been extended to the cases of arbitrary classical Lie algebra. The BGRs obtained by this extended KDT not only covered the standard BGRs, but also included some new BGRs that are called non-standard BGRs [6-10].

The key point for the above-mentioned extended approach is the proposal of weight conservation which is a generalisation of charge conservation [5, 11] (for the details see [12]). Let  $w_a$  ( $a = 1, 2, \dots, N$ ) be the weights of an *N*-dimensional irreducible representation  $\Gamma^{[\lambda]}$  of a classical Lie algebra and  $V^{[\lambda]}$  the corresponding space. A BGR *R* associated with the Lie algebra is defined on the product space  $V^{[\lambda]} \otimes V^{[\lambda]}$  and its matrix element is labelled by the weight pair  $(w_a, w_b)$ , i.e.

$$R_{cd}^{ab} \equiv R_{w_c, w_d}^{w_a, w_b}$$

The weight conservation requires that

$$R_{cd}^{ab} = 0 \quad \text{if } w_a + w_b \neq w_c + w_d. \tag{1}$$

Then *R* can be written as a block diagonal structure

$$R = \text{block diag} (A_1, A_2, \dots, A_k) \tag{2}$$

and the extended KDT can be used to determine each block  $A_i$ . In the following we prove that the above weight conservation structure (2) is satisfied by the BGRs constructed from QG.

From a QG  $U(L)_q$  generated by  $h_{\alpha_i}$ ,  $E_{\alpha_i}$  and  $E_{-\alpha_i}$  that satisfy

$$[h_{\alpha_i}, h_{\alpha_j}] = 0 \quad [E_{\alpha_i}, E_{-\alpha_i}] = \delta_{ij} \quad [h_{\alpha_i}, [h_{\alpha_i}, E_{\alpha_i}]] = \pm A_{ij} E_{\alpha_i} \tag{3}$$

the universal  $R$ -matrix is defined as

$$\hat{R} = \sum_a e_a \otimes e^a \tag{4}$$

where  $\alpha_i$ s are the simple roots of a Lie algebra  $L$  with the Cartan matrix  $A = (A_{ij})$  and  $f = (q^f - q^{-f}) / (q^1 - q^{-1})$  for any operator or number  $f$ .  $e_a$ s are the basis for the subalgebra  $U(L_+)_q$  generated by  $h_{\alpha_i}$  and  $E_{\alpha_i} (i = 1, 2, \dots, l)$  and  $e^a$ 's are the dual basis of the dual  $U(L_+)_q^*$  of  $U(L_+)_q$ , which is an isomorphism of the subalgebra  $U(L_-)_q$  as a Hopf algebra generated by  $h_{-\alpha_i}$  and  $E_{-\alpha_i} (i = 1, 2, \dots, l)$  [13]. For a given matrix representation  $\Gamma^{[\lambda]}$  of  $U(L)$ ,  $R = \Gamma^{[\lambda]} \otimes \Gamma^{[\lambda]}(\hat{R})$  defines a matrix solution of the YBE:

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}. \tag{5}$$

According to Rosso's discussion about the analogue of the PBW theorem and its generalisation [13],  $e_a$  and  $e^a$  take the forms

$$e_a = \dots E_{\alpha_i}^{m_i} \dots E_{\alpha_j}^{m_j} \dots \quad e^a = \dots E_{-\alpha_i}^{m_i} \dots E_{-\alpha_j}^{m_j} \dots \quad m_i, m_j = 0, 1, 2, \dots$$

Then, the universal  $R$ -matrix (4) is rewritten as

$$\hat{R} = \sum_{\dots m_i \dots m_j \dots} \dots E_{\alpha_i}^{m_i} \dots E_{\alpha_j}^{m_j} \dots \otimes \dots E_{-\alpha_i}^{m_i} \dots E_{-\alpha_j}^{m_j} \dots \tag{6}$$

in explicit form. It is easily observed from (6) that  $E_{\alpha_i}$  and  $E_{-\alpha_i}$  appear with the same power  $m_i$ . The matrix element of  $\hat{R}$  satisfies

$$R_{cd}^{ab} = \langle w_a, w_b | \hat{R} | w_c, w_d \rangle \begin{cases} = 0 & \text{if } w_a + w_b \neq w_c + w_d \\ \neq 0 & \text{if } w_a + w_b = w_c + w_d \end{cases} \tag{7}$$

because of the actions of  $E_{\alpha_i}$  and  $E_{-\alpha_i}$  on the weight vectors  $|w_a\rangle$ :

$$E_{\pm\alpha_i}^{m_i} |w_a\rangle = \begin{cases} 0 & \text{if } w_a \pm m_i \alpha_i \text{ is not a root} \\ |w_a \pm m_i \alpha_i\rangle & \text{if } w_a \pm m_i \alpha_i \text{ is a root} \end{cases} \tag{8}$$

where  $|w_a, w_b\rangle = |w_a\rangle \otimes |w_b\rangle$ . It is immediately obvious from the above discussion that the quantum group construction (4) indeed satisfies the weight conservation and the BGRs directly obtained from the proposal of weight conservation include the non-standard BGRs as well as covering the standard ones.

An example of  $su(2)$  can be used to illustrate the above general analysis. In this case the universal  $R$ -matrix is

$$\hat{R} = q^{2(h \otimes h)} \sum_i q^{3/2 i(i-1)} \{ (q - q^{-1})^i / [i] \} E_+^i q^{ih} \otimes E_-^i q^{-ih} \tag{9}$$

where the generators  $h$  and  $E_{\pm}$  of  $U(su(2))_q$  satisfy  $[h, E_{\pm}] = \pm E_{\pm}$  and  $[E_+, E_-] = [2h]$ . For the angular momentum representation with the basis  $|j, m\rangle$  [15], we have

$$\langle m_1, m_2 | \hat{R} | m_3, m_4 \rangle = \begin{cases} 0 & \text{if } m_1 + m_2 \neq m_3 + m_4 \\ \neq 0 & \text{if } m_1 + m_2 = m_3 + m_4. \end{cases} \tag{10}$$

Now, a question naturally arises: can we add some terms  $\Delta \hat{R}$  that still satisfy weight conservation to the standard  $R$ -matrix (4) such that  $\hat{R} = \hat{R} + \Delta \hat{R}$  also give a solution of the YBE? The answer is positive. In fact, we let

$$\Delta \hat{R} = \sum_{m+m'=n+n'} C_{m'n',r}^{mnr} E_+^m E_-^n h^r \otimes E_+^{m'} E_-^{n'} h^{r'} \tag{11}$$

where the coefficients  $C_{m'n'r}^{mnr}$  can be determined by substituting  $R = R + \Delta R$  into the YBE (5) for a given irreducible representation of  $U(\mathfrak{su}(2))_q$  [14]. For the case  $j = \frac{1}{2}$ ,

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we determine

$$\Delta R_{1/2} = -(q^{-3/2} + q^{1/2}) E_- E_+ \otimes E_- E_+. \tag{12}$$

For the case  $j = 1$ ,

$$E_+ = [2]^{1/2} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad E_- = [2]^{1/2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

we determine

$$\begin{aligned} \Delta R_1 = & (\omega q^{-6} - q^2) [2]^{-3} \{ [2]^{-1} (E_-^2 E_+^2 \otimes E_-^2 E_+^2) + E_- E_+^2 \otimes E_- E_+^2 \} + (\omega^2 q^{-4} - 1) [2]^{-3} \\ & \times \{ (E_+ E_- - [2]^{-1} E_+^2 E_-^2) \otimes E_-^2 E_+^2 + E_-^2 E_+^2 \otimes (E_+ E_- - [2]^{-1} E_+^2 E_-^2) \} \\ & + q^{-2} (q^2 - q^{-2}) (1 - q^{-2} \omega) [2]^{-2} E_+^2 \otimes E_-^2 + q^{-1} \{ \omega (q^2 - q^{-2}) (q^{-4} \omega - 1) \}^{1/2} \\ & - (q - q^{-2}) \} [2]^{-3} (E_+^2 E_- \otimes E_-^2 E_+ + E_- E_+^2 \otimes E_+ E_-^2) \\ & + (\omega q^{-2} - 1) [2]^{-2} (E_+ E_- - [2]^{-1} E_+^2 E_-^2) \otimes (E_+ E_- - [2]^{-1} E_+^2 E_-^2) \\ & \omega^3 = 1. \end{aligned} \tag{13}$$

Equations (12) and (13) lead to two non-standard BGRs

$$R'_{1/2} = R_{1/2} + \Delta R_{1/2} \approx q^{-1/2} \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{bmatrix} \tag{14a}$$

$$R'_1 = R_1 + \Delta R_1 = \text{block diag} (A_1, A_2, A_3, A'_2, A'_1) \tag{14b}$$

$$A_1 = q^2 \quad A'_1 = \omega q^{-6} \quad A_2 = \begin{bmatrix} 1 & q - q^{-2} \\ 0 & 1 \end{bmatrix}$$

$$A'_2 = \begin{bmatrix} \omega^2 q^{-4} & q^{-2} (\omega q^{-6} - 1) \\ 0 & \omega^2 q^{-4} \end{bmatrix}$$

$$A_3 = \begin{bmatrix} q^{-2} & q^{-1} \omega \{ (q^2 - q^{-2}) (q^{-4} \omega - 1) \}^{1/2} & (q^2 - q^{-2}) (1 - \omega q^{-4}) \\ 0 & \omega q^{-2} & q^{-1} \omega \{ (q^2 - q^{-2}) (q^{-4} \omega - 1) \}^{1/2} \\ 0 & 0 & q^{-2} \end{bmatrix} \tag{14c}$$

which has been given in [15] and can be verified by the extended KDT to satisfy (5).

The discussion about non-standard BGRs associated with arbitrary classical Lie algebra based on this letter will be published elsewhere in which the explicit construction of representation for QC through the  $q$ -deformed boson realisation [14, 16] has been applied.

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